RAN-22030803010400044

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## M.Sc. (Sem. - I) Examination November - 2025

## Mathematics (Paper - PGMTH - 1044) Advanced Number Theory

**Time: 3 Hours ]** **[ Total Marks: 70**

| **સૂચના : / Instructions** | Seat No.: |
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| (1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.<br>Fill up strictly the above details on your answer book<br>(2) Follow usual notations and conventions.<br>(3) Figures to the right indicate full marks of the corresponding question.<br>(4) Use of non-programmable scientific calculator is permitted. | Student's Signature |

**Q.1. Answer any TWO of the following: (14)**

(1) Prove that any integer of the form $2^k$; $k \geq 3$ do not have a primitive root.

(2) Define primitive root of the integer $n$. If $p$ is an odd prime, then show that $2 \times p^k$ always has a primitive root for every positive integer $k$.

(3) If the integer $a$ has order $k$ modulo $n$, then prove the following:

(a) $a^h \equiv 1\ (mod\ n)$ if and only if $k \mid h$.

(b) Order of $a^h$ modulo $n$ is $\dfrac{k}{\gcd(h,k)}$; $h > 0$.

**Q.2. Answer any TWO of the following: (14)**

(1) State and prove the basic rules of indices which remind you of logarithm.

(2) Let $p$ be an odd prime and gcd $(a, p) = 1$. Then prove that $\left(\dfrac{a}{p}\right) = -1$ if and only if $a^{\frac{p-1}{2}} \equiv -1\ (mod\ p)$.

(3) Prove that any odd prime $p$ has exactly $\dfrac{p-1}{2}$ incongruent quadratic residues namely $1^2, 2^2, \ldots, \left(\dfrac{p-1}{2}\right)^2$. Hence conclude that $p$ has exactly $\dfrac{p-1}{2}$ quadratic non-residues.

**Q.3. Answer any TWO of the following: (14)**

(1) Only state the 'Quadratic Reciprocity law'. Use it to find the value of $(p/q)$ when any one of $p$ or $q$ is (or none of $p$ and $q$ are) of the form $4k+1$. Hence find the value of $\left(\dfrac{-46}{17}\right)$.

(2) Prove that the congruence $x^2 \equiv a\ (mod\ 2^n)$; $n \geq 3$ has a solution if and only if $a \equiv 1\ (mod\ 2^3)$.

(3) For an odd prime $p$, prove that $\left(\dfrac{2}{p}\right) = 1$ if and only if $p \equiv \pm 1\ (mod\ 8)$.

Hence show that $\left(\dfrac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}$.

**Q.4. Answer any TWO of the following: (14)**

(1) In usual notations, prove that $u_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$; $n \geq 1$. Hence show that $u_n = \left[\dfrac{\alpha^n}{\sqrt{5}} + \dfrac{1}{2}\right]$, given that $0 < |\beta| < 1$.

(2) Define: Amicable numbers and explain it using an example. If the three numbers $p = 3 \times 2^{n-1} - 1$, $q = 3 \times 2^{n} - 1$ and $r = 9 \times 2^{2n-1} - 1$ are all primes and $n \geq 2$, then show that $2^n pq$ and $2^n r$ are amicable numbers.

(3) If $n$ is an odd perfect number, then show that $n = p_1^{k_1} p_2^{2j_2} \ldots p_r^{2j_r}$; where the P$i$'s are distinct odd primes and $p_1 \equiv k_1 \equiv 1\ (mod\ 4)$.

**Q.5. Answer any TWO of the following: (14)**

(1) (a) Prove that there are infinitely many primes of the form $4k + 1$.

(b) Use Gauss' Lemma to determine whether 7 is a quadratic residue of 19.

(2) Solve the quadratic congruence $3x^2 + 9x + 7 \equiv 0\ (mod\ 13)$.

(3) (a) State and prove Cassini's identity for Fibonacci numbers.

(b) Find the sum of (i) first $n$ Fibonacci numbers

(ii) first $n$ Fibonacci numbers with even subscripts.

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